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On a Linear Programming Type Theorem in Locally Convex Linear Topological Space.

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1. Abstract

The following is well known as the fundamental theorem of linear programming: In an n-dimensional Euclidean space, the set of feasible solutions in the problem of linear programming forms a convex polyhedron K, and if K is bounded, any linear functional f(x) attains the maximum on K at a vertex of K. Here we prove an analogous type theorem in a locally convex linear topological space and show an application.

2. Definitions

Let X be a linear space and $K \subset X$. A non-void subset E of K is said to be an **extremal subset** of K, if a proper convex combination $\mu x+\nu y$ ($\mu>0$, $\nu>0$, $\mu+\nu=1$) of two points x, y of K is in E only if both x and y are in E. An extremal subset of K consisting of just one point is called an **extremal point** of K. For example, in a 3-dimensional Euclidean space, the vertices, sides and faces of a polyhedron form an extremal subset of the polyhedron, but only the vertices are extremal points of the polyhedron. A subset K of X is said to be **convex,** if $\mu x+\nu y \in K$ whenever $\mu \geqq 0$, $\nu \geqq 0$, $\mu+\nu=1$ and x, y $\in$ K. We denote the set of extremal points of K by $\mathbf{K}^e$. A linear space is said to be a **linear topological space,** if it is a Hausdorff space and the operations x-y and αx are continuous, where α denotes a scalar. A linear topological space is said to be **locally convex,** if it possesses a base for its topology consisting of convex sets. Let p(x) be a real valued functional on a linear space X. We say that p(x) is **convex,** if $p(\mu x+\nu y) \leqq \mu p(x)+\nu p(y)$ whenever $\mu \geqq 0$, $\nu \geqq 0$, $\mu+\nu=1$ and x, y $\in$ X.

3. Theorem

The following lemma is known.

Lemma 1. [2] A non-void compact set in a locally convex linear topological space has an extremal point.

Theorem 2. Let X be a locally convex linear topological space, K be a compact subset of X, and p(x) be a real valued continuous convex functional on X. Then p(x) attains the maximum on K at an extremal point of K.

Proof. Let $M=\sup_{x\in K} p(x)$ and $E=\{x\in K \mid p(x)=M\}$.

Since K is comact and p(x) is continuous, p(x) attains the maximum on K at a point of K. So $E\fallingdotseq\phi$.

Suppose $p(\mu x+\nu y)=M$ ($\mu>0$, $\nu>0$, $\mu+\nu=1$ and $x,y\in K$).

If $p(x)<M$, then $M=p(\mu x+\nu y)\leqq \mu p(x)+\nu p(y) <\mu M+\nu p(y)\leqq \mu M+\nu M=(\mu+\nu)M=M$. This contradiction implies $p(x)=M$. Analogously we get $p(y)=M$. Therefore

E is an extremal subset of K. Let $G=\{x\in X \mid p(x)<M\}$ and a be an arbitrary element of G. By the continuity of p(x), there exists an open neighborhood U_a such that $|p(U_a)-p(a)|<\varepsilon$ for $\varepsilon=\frac{1}{2}(M-p(a))>0$. Thus $p(U_a)<p(a)+\varepsilon=M-\varepsilon<M$. This shows $U_a\subset G$. Since $E=K\cap G^c$, where G^c is the complent of G in X, E is a closed subset of K. Therefore E is compact. By Lemma 1, there exists an element $x_o\in E^e$. It follows by $x_o\in E$ that $\mu x+\nu y=x_o$ ($\mu>0$, $\nu>0$, $\mu+\nu=1$ and $x, y\in K$) implies $x, y\in E$.

Thus we get $x=y=x_o$ by the definition of E^e. Therefore x_o is an extremal point of K and $p(x_o)=M$. q.e.d.

Let f(x) be a continuous linear functional on X. Then $|f(x)|$ is a real valued continuous convex functional on X. Therefore Theorem 2 shows the following

Corollary 3. Let X be a locally convex linear topological space, K be a compact subset of X, and f(x) be a contnuous linear functional on X. Then $|f(x)|$ attains the maximum on K at an extremal point of K.

If f(x) is a real valued linear functional, then f(x) and -f(x) are convex functionals. So Theorem 2 holds for f(x) and -f(x). Therefore we get the following

Corollary 4. Let X be a locally convex linear topological space, K be a compact subset of X, and f(x) be a real valued continuous linear functional on X. Then f(x) attains the maximum on K at an extremal point of K, and also attains the minimum on K at an extremal point of K.

4. Applicalion

Let R^2 be the 2-dimensional Euclidean space and L be the space of all 2×2 matrices of real numbers.

By the norm $\|A\| = \sup_{x \in R^2} \frac{\|Ax\|}{\|x\|}$ and the $*$-operation

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longrightarrow A^* = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$, L is a normed $*$-algebra

with the identity $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

As an application of Corollary 3, we get the norm of real valued linear functionals concretely.

Theorem 5. Let φ be a real valued linear functional on L.

and let $\varphi_{11} = \varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)$, $\varphi_{12} = \varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$, $\varphi_{21} = \varphi\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)$, $\varphi_{22} = \varphi\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$.

Then $\|\varphi\| = \sup_{A \in L} \frac{|\varphi(A)|}{\|A\|} = \sqrt{\varphi_{11}^2 + \varphi_{12}^2 + \varphi_{21}^2 + \varphi_{22}^2 + 2\,|J|}$,

where $J = \begin{vmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{vmatrix}$.

Proof. Let $K = \{A \in L \mid \|A\| \leqq 1\}$. Since K is compact and φ is automatically continuous, $|\varphi(A)|$ attains the maximum on K at an exremal point of K by Corollary 3. Therefore $\|\varphi\| = \sup_{A \in L} \frac{|\varphi(A)|}{\|A\|} = \sup_{A \in K} |\varphi(A)|$

$$= \max_{A \in K^e} |\varphi(A)|.$$

$K^e = \{U \in L \mid UU^* = U^*U = I\}$ is known (1) (*). Such U's are only two types: $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ for arbitrary real number θ.

Thus $|\varphi(U)| = |\cos\theta \cdot \varphi_{11} + \sin\theta \cdot \varphi_{12} \pm \sin\theta \cdot \varphi_{21} \mp \cos\theta \cdot \varphi_{22}|$

$$= |\cos\theta(\varphi_{11} \mp \varphi_{22}) + \sin\theta(\varphi_{12} \pm \varphi_{21})|$$

$$= \sqrt{p^2 + q^2} \cdot |\cos(\theta - \alpha)|,$$

(*) L is a real algebra, so a slight modification is necessary.

where $p=\varphi_{11}\mp\varphi_{22}$, $q=\varphi_{12}\pm\varphi_{21}$, $\tan\alpha=\frac{q}{p}$.

Therefore $\max|\varphi(U)|=\max\sqrt{p^2+q^2}$

$$=\max\sqrt{\varphi_{11}^2+\varphi_{12}^2+\varphi_{21}^2+\varphi_{22}^2\pm 2J}$$

$$=\sqrt{\varphi_{11}^2+\varphi_{12}^2+\varphi_{21}^1+\varphi_{22}^2+2|J|},$$

where $J=\begin{vmatrix}\varphi_{11} & \varphi_{12}\\ \varphi_{21} & \varphi_{22}\end{vmatrix}$.　　q. e. d.

References

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局所凸な線型位相空間における線型計画型の定理とそのひとつの応用

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n次元ユークリッド空間における線型計画の問題において許容解の集合が凸多面体をなし，その凸多面体が有界ならば与えられた実線型汎函数はその多面体の頂点で最大値をとることは線型計画の基本定理として知られている。

この小論では局所凸な線型位相空間において線型計画の基本定理に相当する定理を証明し，そのひとつの応用を示す。